

Factorisation structures of algebras and coalgebras

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Abstract

We consider the factorisation problem for bialgebras: when a bialgebra K factorises as $K = HL$, where H and L are algebras and coalgebras (but not necessarily bialgebras). Given two maps $R : H \otimes L \rightarrow L \otimes H$ and $W : L \otimes H \rightarrow H \otimes L$, we introduce a product $L {}_{W \bowtie_R} H$, and we give necessary and sufficient conditions for $L {}_{W \bowtie_R} H$ to be a bialgebra. It turns out that K factorises as $K = HL$ if and only if $K \cong L {}_{W \bowtie_R} H$ for some maps R and W . As examples of this product we recover constructions introduced by Majid ([13]) and Radford ([20]). Also some of the pointed Hopf algebras that were recently constructed by Beattie, Dăscălescu and Grünenfelder [2] appear as special cases.

Introduction

The factorisation problem for a "structure" (group, algebra, coalgebra, bialgebra) can be roughly stated as follows: in which conditions an object X can be written as a product of two subobjects A and B which have minimal intersection (for example $A \cap B = \{1_X\}$ in the group case). A related problem is that of the construction of a new object (let us denote it by AB) out of the objects A and B . In the constructions of this type existing in the literature ([20], [27], [13]), the object AB factorises into A and B . This is the case - to give an example - with Majid's double crossed product of Hopf algebras. Moreover, whenever a Hopf algebra factorises in a natural way into two sub-Hopf algebras, it is likely to be a double crossed product ([15, Thm. 7.2.3]). This examples include the quantum double of V.G. Drinfel'd, and lead also to natural generalisations of it on a pairing or skew-pairing of bialgebras.

The simplest example of algebra factorisation is the tensor product of two k -algebras or, more general, the tensor product of two algebras A and B in a braided monoidal category. If A and B

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are such algebras, the multiplication is then given by the formula

$$m_{A\#B} = (m_A \otimes m_B) \circ (I_A \otimes R_{A,B} \otimes I_B) \quad (1)$$

where R is the braiding on the category. In order to apply (1) to two particular algebras A and B , we do not need a braiding on the whole category, it suffices in fact to have a map $R : B \otimes A \rightarrow A \otimes B$. This new algebra $A\#_R B$ will be called a smash product, if it is associative with unit $1_A\#1_B$. We will work in the category of vector spaces over a field k , and give necessary and sufficient conditions for R to define a smash product. The smash product can be determined completely by a universal property, and it will also turn out that any algebra which factorises into A and B is isomorphic with such a smash product. Therefore, we recover in this way several constructions that appeared earlier in the literature as special cases of the smash product.

The construction can be dualized, leading to the definition of the smash coproduct of two coalgebras C and D (Section 3). The main result of this note (Section 4) is the fact that we can combine the two constructions, and this leads to the definition of the smash biproduct of two vector space H and L that are at once algebras and coalgebras (but not necessarily bialgebras). The smash biproduct can be also characterized by what we called bialgebra factorisation structures (Theorem 4.4). Adopting this point of view for the constructions due to Majid [13] and Radford [20], we find that their constructions are characterized by special classes of bialgebra factorisations. We also prove that some of the pointed Hopf algebras that Beattie, Dăscălescu and Grünenfelder [2] constructed using iterated Ore extensions (including classical examples like Sweedler's four dimensional Hopf algebra) can be viewed as smash biproducts (Theorem 5.1).

As we indicated at the beginning of this introduction, the smash biproduct can be defined in an arbitrary monoidal category. Bernhard Drabant kindly informed us that this general biproduct has been introduced recently by Bespalov and Drabant in the forthcoming [3], where it is called a cross product bialgebra.

1 Notations

Let k be a field. For two vector spaces V and W and a k -linear map $R : V \otimes W \rightarrow W \otimes V$ we write

$$R(v \otimes w) = \sum R_w \otimes R_v$$

for all $v \in V, w \in W$. Using this notation, the k -linear map

$$(R \otimes I_V)(I_V \otimes R) : V \otimes V \otimes W \rightarrow W \otimes V \otimes V$$

can be denoted as follows (we write $R = r$):

$$(R \otimes I_V)(I_V \otimes R)(v_1 \otimes v_2 \otimes w) = \sum r(Rw) \otimes r v_1 \otimes R v_2$$

for all $v_1, v_2 \in V, w \in W$.

Let A be a k -algebra. $m_A : A \otimes A \rightarrow A$ will be the multiplication map on A and 1_A the unit of A . For a k -coalgebra C , $\Delta_C : C \rightarrow C \otimes C$ will be the comultiplication and $\varepsilon_C : C \rightarrow k$ the augmentation map.

2 The factorisation problem for algebras

Let H, K and G be groups. We say that G factorises as $G = HK$ if H, K are subgroups of G and $H \cap K = \{1_G\}$. The problem of group factorisations was considered before in [27] where it led to the

definition of bismash products of Hopf algebras. One of the results regarding group factorisations is that whenever G factorises as $G = HK$, (H, K) is a matched pair of groups and $G \cong H \bowtie K$, the product associated with this pair. In order to prove similar results at the algebra level, we need the following definitions.

Definition 2.1 *Let A , B and X be k -algebras with unit. We say that X factorises as $X = AB$ if there exists algebra morphisms*

$$A \xrightarrow{i_A} X \xleftarrow{i_B} B$$

such that the k -linear map

$$\zeta = m_X \circ (i_A \otimes i_B) : A \otimes B \rightarrow X$$

is an isomorphism of vector spaces.

Let A and B be associative k -algebras with unit, and consider a k -linear map $R : B \otimes A \rightarrow A \otimes B$. By definition $A \#_R B$ is equal to $A \otimes B$ as a k -vector space with multiplication given by the formula

$$m_{A \#_R B} = (m_A \otimes m_B)(I_A \otimes R \otimes I_B) \quad (2)$$

or

$$(a \#_R b)(c \#_R d) = \sum a^R c \#_R^R b d \quad (3)$$

for all $a, c \in A$, $b, d \in B$.

Definition 2.2 *Let A and B be k -algebras with unit, and $R : B \otimes A \rightarrow A \otimes B$ a k -linear map. If $A \#_R B$ is an associative k -algebra with unit $1_A \# 1_B$, we call $A \#_R B$ a smash product.*

Remark 2.3 Let R be a braiding on the category \mathcal{M}_k of k -vector spaces. Then we have maps $R_{X,Y} : Y \otimes X \rightarrow X \otimes Y$ for all vector spaces X and Y and our smash product $A \#_{R_{A,B}} B$ is the usual product in the braided category $(\mathcal{M}_k, \otimes, k, R_{X,Y})$ (see [14]).

Our definition has a local character. In fact, given two algebras A and B , one can sometimes compute explicitly all the maps R that make $A \#_R B$ into a smash product (see Example 2.12, 3).

Examples 2.4 1) Let $R = \tau_{B,A} : B \otimes A \rightarrow A \otimes B$ be the switch map. Then $A \#_R B = A \otimes B$ is the usual tensor product of A and B .

2) Let G be a group acting on the k -algebra A . This means that we have a group homomorphism $\sigma : G \rightarrow \text{Aut}_k(A)$. Writing $\sigma(g)(a) = {}^g a$, we find a k -linear map

$$R : kG \otimes A \rightarrow A \otimes kG, \quad R(g \otimes a) = {}^g a \otimes g$$

and $A \#_R k[G] = A *_{\sigma} G$ is the usual skew group algebra.

3) More generally, let H be a Hopf algebra, A a left H -module algebra and D be a left H -comodule algebra. Let

$$R : D \otimes A \rightarrow A \otimes D, \quad R(d \otimes a) = \sum d_{<-1>} \cdot a \otimes d_{<0>}.$$

Then $A \#_R D = A \# D$ is Takeuchi's smash product [28]. For $D = H$, we obtain the usual smash product $A \# H$ defined in Sweedler's book [24].

4) Let (G, H) be a matched pair of groups and $H \bowtie G$ the product associated to this pair (see [27]). Write

$$\begin{aligned} G \times H &\rightarrow H & (g, h) &\mapsto g \cdot h \\ G \times H &\rightarrow G & (g, h) &\mapsto g^h \end{aligned}$$

for the respective group actions, and define

$$R : kG \otimes kH \rightarrow kH \otimes kG, \quad R(g \otimes h) = g \cdot h \otimes g^h$$

for all $g \in G$ and $h \in H$. Then $kH \#_R kG = k[H \bowtie G]$. In Example 2.12, 4), we will construct an example of a smash product $kH \#_R kG$ which is not of the form $k[H \bowtie G]$.

Definition 2.5 Let A and B be k -algebras and $R : B \otimes A \rightarrow A \otimes B$ a k -linear map. R is called *left normal* if

$$(LN) \quad R(b \otimes 1_A) = 1_A \otimes b$$

for all $b \in B$. R is called *right normal* if

$$(RN) \quad R(1_B \otimes a) = a \otimes 1_B$$

for all $a \in B$. We call R *normal* if R is left and right normal.

The problem of algebra factorisations was studied before in the first part of the proof of Theorem 7.2.3 in [15]. For reader's convenience we present in the Theorem 2.6 and Theorem 2.11 the detailed proofs of the results contained there. In the following Theorem, we give necessary and sufficient conditions for $A \#_R B$ to be a smash product.

Theorem 2.6 Let A, B be two algebras and let $R : B \otimes A \rightarrow A \otimes B$ be a k -linear map. The following statements are equivalent

1. $A \#_R B$ is a smash product.
2. The following conditions hold:
 - (N) R is normal;
 - (O) the following octogonal diagram is commutative

$$\begin{array}{ccccc} B \otimes A \otimes A \otimes B & \xrightarrow{I_B \otimes m_A \otimes I_B} & B \otimes A \otimes B & \xrightarrow{R \otimes I_B} & A \otimes B \otimes B \\ \uparrow I_B \otimes I_A \otimes R & & & & \downarrow I_A \otimes m_B \\ B \otimes A \otimes B \otimes A & & & & A \otimes B \\ \downarrow R \otimes I_B \otimes I_A & & & & \uparrow m_A \otimes I_B \\ A \otimes B \otimes B \otimes A & \xrightarrow{I_A \otimes m_B \otimes I_A} & A \otimes B \otimes A & \xrightarrow{I_A \otimes R} & A \otimes A \otimes B \end{array}$$

3. The following conditions hold:

(N) R is normal;

(P) the following two pentagonal diagrams are commutative:

(P1)

$$\begin{array}{ccccc}
 B \otimes B \otimes A & \xrightarrow{m_B \otimes I_A} & B \otimes A & \xrightarrow{R} & A \otimes B \\
 \downarrow I_B \otimes R & & & & \uparrow I_A \otimes m_B \\
 B \otimes A \otimes B & \xrightarrow{R \otimes I_B} & A \otimes B \otimes B & &
 \end{array}$$

(P2)

$$\begin{array}{ccccc}
 B \otimes A \otimes A & \xrightarrow{I_B \otimes m_A} & B \otimes A & \xrightarrow{R} & A \otimes B \\
 \downarrow R \otimes I_A & & & & \uparrow m_A \otimes I_B \\
 A \otimes B \otimes A & \xrightarrow{I_A \otimes R} & A \otimes A \otimes B & &
 \end{array}$$

Proof: 3) \Rightarrow 1) follows from [29, Prop. 2.2 and 2.3].

1) \Leftrightarrow 2) An easy computation shows that $1_A \# 1_B$ is a right (resp. left) unit of $A \#_R B$ if and only if R is right (resp. left) normal.

Let us prove that the multiplication $m_{A \#_R B}$ is associative if and only if the octogonal diagram (O) is commutative. Using the notation introduced in Section 1, we find that the commutativity of the diagram (O) is equivalent to the following formula

$$\sum r(a_2^R a_3) \otimes {}^r b_1^R b_2 = \sum {}^R a_2^r a_3 \otimes {}^r ({}^R b_1 b_2) \quad (4)$$

for all $a_2, a_3 \in A$, $b_1, b_2 \in B$ (where $r = R$). Now for all $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$, we have that

$$(a_1 \#_R b_1)((a_2 \#_R b_2)(a_3 \#_R b_3)) = \sum a_1^r (a_2^R a_3) \#_R {}^r b_1^R b_2 b_3 \quad (5)$$

and

$$((a_1 \#_R b_1)(a_2 \#_R b_2))(a_3 \#_R b_3) = \sum a_1^R a_2^r a_3 \#_R {}^r ({}^R b_1 b_2) b_3. \quad (6)$$

and the associativity of the multiplication follows.

Conversely, if $m_{A \#_R B}$ is associative, then (4) follows after we take $a_1 = 1_A$ and $b_3 = 1_B$ in (5-6).

2) \Rightarrow 3) Suppose that (O) is commutative, or, equivalently, (4) holds. Taking $a_2 = 1$ in (4), we find, taking the normality of R into account,

$$\sum {}^r ({}^R a_3) \otimes {}^r b_1^R b_2 = \sum {}^r a_3 \otimes {}^r (b_1 b_2) \quad (7)$$

and this is equivalent to commutativity of (P1). Taking $b_2 = 1$, we find

$$\sum {}^r (a_2 a_3) \otimes {}^r b_1 = \sum {}^R a_2 \otimes {}^r a_3 \otimes {}^r ({}^R b_1) \quad (8)$$

and this is equivalent to commutativity of (P1).

3) \Rightarrow 2) Assume that (P1) and (P2) are commutative. Applying successively (8) and (7), we find

$$\begin{aligned}
 \sum {}^r (a_2^R a_3) \otimes {}^r b_1^R b_2 &= \sum {}^R a_2^r ({}^R a_3) \otimes {}^r ({}^R b_1)^R b_2 \\
 &= \sum {}^R a_2^r a_3 \otimes {}^r ({}^R b_1 b_2)
 \end{aligned}$$

and this proves (4) and the commutativity of (O). \square

Theorem 2.6 leads us to the following definition.

Definition 2.7 *Let A and B be k -algebras, and $R : B \otimes A \rightarrow A \otimes B$ a k -linear map. R is called left (resp. right) multiplicative if (P1) (resp. (P2)) is commutative, or, equivalently, if (7) (resp. (8)) holds. R is called multiplicative if R is at once left and right multiplicative.*

Remarks 2.8 1) The conditions (7-8) for the commutativity of the two pentagonal diagrams will turn out to be useful for computing explicit examples. These conditions already appeared in [19], [26], and Theorem 2.6 assures us that they can be replaced by one single condition (4) if R is normal. A map R for which the diagram (O) is commutative could be called octogonal.

2) Let $R : B \otimes A \rightarrow A \otimes B$ be normal. Then R is left multiplicative if and only if

$$((1_A \#_R b)(1_A \#_R d))(a \#_R 1_B) = (1_A \#_R b)((1_A \#_R d)(a \#_R 1_B)),$$

for all $a \in A$ and $b, d \in B$. Similarly, R is right multiplicative if and only if

$$(1_A \#_R b)((a \#_R 1_B)(c \#_R 1_B)) = ((1_A \#_R b)(a \#_R 1_B))(c \#_R 1_B),$$

for all $a, c \in A$ and $b \in B$.

Examples 2.9 1) Let H be a bialgebra and assume that A and B are algebras in the category $H\mathcal{M}$. This means that A and B are left H -module algebras. Take $x = \sum x^1 \otimes x^2 \in H \otimes H$ and consider the map

$$R = R_x : B \otimes A \rightarrow A \otimes B, \quad R(b \otimes a) = \sum x^2 \cdot a \otimes x^1 \cdot b$$

If x satisfies the conditions (QT1-QT4) in the definition of a quasitriangular bialgebra (see e.g. [18],[21]), then R_x is normal and multiplicative. Observe that we do not need that (H, x) is quasitriangular, since we do not require (QT5).

2) Now let H be a bialgebra, and let A and B be H -comodule algebras, or algebras in the category \mathcal{M}^H . For a k -linear map $\sigma : H \otimes H \rightarrow k$, we define $R = R_\sigma : B \otimes A \rightarrow A \otimes B$ by

$$R(b \otimes a) = \sum \sigma(a_{<1>} \otimes b_{<1>}) a_{<0>} \otimes b_{<0>}$$

If σ satisfies conditions (CQT1-CQT4) in the definition of a coquasitriangular bialgebra (cf. [18]), then $R = R_\sigma$ is normal and multiplicative. Observe that it is not necessary that (H, σ) is coquasitriangular.

Remarks 2.10 1) Let $A \#_R B$ be a smash product. Then the maps

$$i_A : A \rightarrow A \#_R B \quad \text{and} \quad i_B : B \rightarrow A \#_R B$$

defined by

$$i_A(a) = a \#_R 1_B \quad \text{and} \quad i_B(b) = 1_A \#_R b$$

are algebra maps. Moreover

$$a \#_R b = (a \#_R 1_B)(1_A \#_R b)$$

for all $a \in A, b \in B$.

2) If $A \#_R B$ is a smash product, then the map

$$\zeta = m_{A \#_R B} \circ (i_A \otimes i_B) : A \otimes B \rightarrow A \#_R B, \quad \zeta(a \otimes b) = a \#_R b$$

is an isomorphism of vector spaces. R can be recovered from ζ by the formula

$$R = \zeta^{-1} \circ m_{A \#_R B} \circ (i_B \otimes i_A)$$

This last remark leads us to the following description of the smash product.

Theorem 2.11 *Let A, B and X be k -algebras. The following conditions are equivalent.*

- 1) *There exists an algebra isomorphism $X \cong A \#_R B$, for some $R : B \otimes A \rightarrow A \otimes B$;*
- 2) *X factorises as $X = AB$.*

Proof: 1) \Rightarrow 2) follows from Remark 2.10.

2) \Rightarrow 1) Suppose there exist algebra morphisms

$$A \xrightarrow{i_A} X \xleftarrow{i_B} B$$

such that the k -linear map

$$\zeta = m_X \circ (i_A \otimes i_B) : A \otimes B \rightarrow X$$

is an isomorphism of vector spaces. Consider

$$R = \zeta^{-1} \circ m_X \circ (i_B \otimes i_A) : B \otimes A \rightarrow A \otimes B$$

We will prove that R is normal, multiplicative and that $\zeta : A \#_R B \rightarrow X$ is an algebra isomorphism.

1) R is left normal. We have to show that $R(b \otimes 1_A) = 1_A \otimes b$, for all $b \in B$, or, equivalently,

$$(\zeta \circ R)(b \otimes 1_A) = \zeta(1_A \otimes b)$$

This follows easily from the following computations:

$$(\zeta \circ R)(b \otimes 1_A) = (m_X \circ (i_B \otimes i_A))(b \otimes 1_A) = i_B(b)i_A(1_A) = i_B(b)$$

and

$$\zeta(1_A \otimes b) = (m_X \circ (i_A \otimes i_B))(1_A \otimes b) = i_A(1_A)i_B(b) = i_B(b),$$

In a similar way, we prove that R is right normal.

2) R is multiplicative. To this end, it suffices to show that

$$(a \#_R b)(c \#_R d) = \zeta^{-1}(\zeta(a \#_R b)\zeta(c \#_R d)), \quad (9)$$

for all $a, c \in A, b, d \in B$. Indeed, (9) means that the multiplication $m_{A \#_R B}$ on $A \#_R B$ can be obtained by translating the multiplication on X using ζ , and this implies that the multiplication on $A \#_R B$ is associative, and that ζ is an algebra homomorphism. (9) is equivalent to

$$\zeta((a \#_R b)(c \#_R d)) = \zeta(a \#_R b)\zeta(c \#_R d) \quad (10)$$

Now $\zeta(a \#_R b)\zeta(c \#_R d) = i_A(a)i_B(b)i_A(c)i_B(d)$. Take $x = i_B(b)i_A(c) \in X$, and write

$$\zeta^{-1}(x) = \sum x_A \otimes x_B \in A \otimes B$$

We easily compute that

$$\begin{aligned}\zeta((a \#_R b)(c \#_R d)) &= \sum i_A(ax_A)i_B(x_Bd) \\ &= i_A(a) \left(\sum i_A(x_A)i_B(x_B) \right) i_B(c)\end{aligned}$$

Now

$$\begin{aligned}\sum i_A(x_A)i_B(x_B) &= (m_X \circ (i_A \otimes i_B) \circ \zeta^{-1}) \circ (m_X \circ (i_B \otimes i_A))(b \otimes c) \\ &= (m_X \circ (i_B \otimes i_A))(b \otimes c) \\ &= i_B(b)i_A(c)\end{aligned}$$

as $m_X \circ (i_A \otimes i_B) \circ \zeta = I$. This proves (10) and completes our proof. \square

Examples 2.12 1) Take a k -algebra A , and let $B = k[t]$ be the polynomial ring in one variable. Consider two k -linear maps $\alpha : A \rightarrow A$ and $\delta : A \rightarrow A$, and define $R : B \otimes A \rightarrow A \otimes B$ in such a way that R is right normal and left multiplicative, and

$$R(t \otimes a) = \alpha(a) \otimes t + \delta(a) \otimes 1$$

for all $a \in A$. This can be done in a unique way. Moreover, R is left normal if and only if $\alpha(1_A) = 1_A$ and $\delta(1_A) = 0$, and R is right multiplicative if and only if α is an algebra morphism and δ is an α -derivation. If R is normal and multiplicative, then $A \#_R B \cong A[t, \alpha, \delta]$, the Ore extension associated to α and δ .

2) Take $a, b \in k$, and let $A = k[X]/(X^2 - a)$, $B = k[X]/(X^2 - b)$. We write i, j for the images of X in respectively A and B , and define $R : B \otimes A \rightarrow A \otimes B$ such that R is normal and

$$R(j \otimes i) = -i \otimes j$$

Then $A \#_R B = {}^a k^b$ is nothing else then the generalized quaternion algebra.

3) Let C_2 be the cyclic group of two elements. We will describe all the smash products of the type $kC_2 \#_R kC_2$. We write $A = kC_2 = k[a]$ and $B = kC_2 = k[b]$ for respectively the first and second factor. If $R : B \otimes A \rightarrow A \otimes B$ is normal, then R is completely determined by

$$R(b \otimes a) = \alpha(a \otimes b) + \beta(a \otimes 1) + \gamma(1 \otimes b) + \delta(1 \otimes 1)$$

with $\alpha, \beta, \gamma, \delta \in k$. It is easy to check that R is multiplicative if and only if the following conditions are satisfied:

$$\begin{aligned}2\alpha\beta &= 0, & \alpha^2 + \beta^2 &= 1, & \alpha\delta + \beta\gamma + \delta &= 0, & \alpha\gamma + \beta\delta + \gamma &= 0 \\ 2\alpha\gamma &= 0, & \alpha\delta + \beta\gamma + \delta &= 0, & \alpha^2 + \gamma^2 &= 1, & \alpha\beta + \gamma\delta + \beta &= 0\end{aligned}$$

An elementary computation shows that we have only the following possibilities for the map R :

(a) If $\text{Char}(k) = 2$, then

$$(i) \quad R(b \otimes a) = a \otimes b + \delta(1 \otimes 1),$$

$$(ii) \quad R(b \otimes a) = (\beta + 1)(a \otimes b) + \beta(a \otimes 1) + \beta(1 \otimes b) + \beta(1 \otimes 1);$$

(b) If $\text{Char}(k) \neq 2$, then

- (i) $R(b \otimes a) = a \otimes b$,
- (ii) $R(b \otimes a) = -(a \otimes b) + \delta(1 \otimes 1)$,
- (iii) $R(b \otimes a) = (a \otimes 1) + (1 \otimes b) - (1 \otimes 1)$,
- (iv) $R(b \otimes a) = (a \otimes 1) - (1 \otimes b) + (1 \otimes 1)$,
- (v) $R(b \otimes a) = -(a \otimes 1) + (1 \otimes b) + (1 \otimes 1)$,
- (vi) $R(b \otimes a) = -(a \otimes 1) - (1 \otimes b) - (1 \otimes 1)$.

We can show that the algebras given by the last four maps are isomorphic, thus our construction gives us only the following algebras:

(a) If $\text{Char}(k) = 2$, then

- (i) $A \#_R B = k\langle a, b | a^2 = b^2 = 1, ab + ba = q \rangle$, where $q \in k$,
- (ii) $A \#_R B = k\langle a, b | a^2 = b^2 = 1, ba = (q + 1)ab + qa + qb + q \rangle$, where $q \in k$;

(b) If $\text{Char}(k) \neq 2$, then

- (i) $A \#_R B = k\langle a, b | a^2 = b^2 = 1, ab = ba \rangle$,
- (ii) $A \#_R B = k\langle a, b | a^2 = b^2 = 1, ab + ba = q \rangle$, where $q \in k$,
- (iii) $A \#_R B = k\langle a, b | a^2 = b^2 = 1, ba = a + b - 1 \rangle$.

4) As a special case of the foregoing example, consider the normal map R defined by

$$R(b \otimes a) = 1_A \otimes 1_B - a \otimes b$$

R is multiplicative, and

$$A \#_R B = k\langle a, b | a^2 = b^2 = 1, ab + ba = 1 \rangle$$

is a four dimensional noncommutative k -algebra. Thus there exists no group G such that

$$kC_2 \#_R kC_2 \cong kG,$$

and, in particular, there exists no matched pair on (C_2, C_2) such that $kC_2 \#_R kC_2 \cong k[C_2 \bowtie C_2]$.

5) Let $A = k\langle x | x^2 = 0 \rangle \cong k[X]/(X^2)$ and $B = kC_2$ with $C_2 = \langle g \rangle$. Take the unique normal map $R : B \otimes A \rightarrow A \otimes B$ such that

$$R(g \otimes x) = -x \otimes g$$

Then $A \#_R B = k\langle g, x | g^2 = 1, x^2 = 0, gx + xg = 0 \rangle$ is Sweedler's four dimensional Hopf algebra, considered as an algebra.

Our next aim is to show that our smash product is determined by a universal property.

Proposition 2.13 *Consider two k -algebras A and B , and let $R : B \otimes A \rightarrow A \otimes B$ be a normal and multiplicative map. Given a k -algebra X , and algebra morphisms $u : A \rightarrow X$, $v : B \rightarrow X$ such that*

$$m_X(v \otimes u) = m_X(u \otimes v)R, \tag{11}$$

we can find a unique algebra map $w : A \#_R B \rightarrow X$ such that the following diagram commutes

$$\begin{array}{ccc}
 & A \#_R B & \\
 i_A \nearrow & \downarrow w & \nwarrow i_B \\
 A & & B \\
 u \searrow & & \swarrow v \\
 & X &
 \end{array}$$

Proof: Assume that w satisfies the requirements of the Proposition. Then

$$w(a \#_R b) = w((a \#_R 1_B)(1_A \#_R b)) = (w \circ i_A)(a)(w \circ i_B)(b) = u(a)v(b),$$

and this proves that w is unique. The existence of w can be proved as follows: define $w : A \#_R B \rightarrow X$ by

$$w(a \#_R b) = u(a)v(b)$$

Then

$$w((a \#_R b)(c \#_R d)) = \sum u(a)u({}^R c)v({}^R b)v(d)$$

and

$$w(a \#_R b)w(c \#_R d) = u(a)v(b)u(c)v(d)$$

and it follows from (11) that w is an algebra map. The commutativity of the diagram is obvious. \square

3 The coalgebra case

The results of Section 2 can be dualized to the coalgebra case. We omit the proofs since they are dual analogs of the corresponding proofs in Section 2.

Definition 3.1 Let C , D and Y be k -coalgebras with counit. We say that Y factorises as $Y = CD$ if there exists coalgebra morphisms

$$C \xleftarrow{p_C} Y \xrightarrow{p_D} D$$

such that the k -linear map

$$\eta : Y \rightarrow C \otimes D, \quad \eta = (p_C \otimes p_D)\Delta_Y$$

is an isomorphism of vector spaces.

Let C and D be two k -coalgebras, and consider a k -linear map

$$W : C \otimes D \rightarrow D \otimes C, \quad W(c \otimes d) = \sum {}^W d \otimes {}^W c$$

Let $C \bowtie_W D$ be equal to $C \otimes D$ as a k -vector space, but with comultiplication given by

$$\Delta_{C \bowtie_W D} = (I_C \otimes W \otimes I_D)(\Delta_C \otimes \Delta_D) \tag{12}$$

or

$$\Delta_{C \bowtie_W D}(c \bowtie d) = \sum (c_{(1)} \bowtie {}^W d_{(1)}) \otimes ({}^W c_{(2)} \bowtie d_{(2)}) \tag{13}$$

Definition 3.2 With notation as above, $C \bowtie_W D$ is called a smash coproduct if the comultiplication (12) is coassociative, with counit map $\varepsilon_{C \bowtie_W D}(c \bowtie d) = \varepsilon_C(c)\varepsilon_D(d)$.

Examples 3.3 1) Let $W = \tau_{C,D} : C \otimes D \rightarrow D \otimes C$ be the switch map. Then $C \bowtie_W D = C \otimes D$ is the usual tensor product of coalgebras.

2) Let H be a bialgebra, C a right H -module coalgebra and D a right H -comodule coalgebra, with coaction $\rho_D : D \rightarrow D \otimes H$, $\rho_D = \sum d_{<0>} \otimes d_{<1>} \in D \otimes H$. Let

$$W : C \otimes D \rightarrow D \otimes C, \quad W(c \otimes d) = \sum d_{<0>} \otimes c \cdot d_{<1>}$$

Then $C \bowtie_W D = C \bowtie D$ is Molnar's smash coproduct [17].

3) If C and D are finite dimensional, then W^* can be viewed as a map $W^* : D^* \otimes C^* \rightarrow C^* \otimes D^*$, after we make the identification $(C \otimes D)^* \cong C^* \otimes D^*$. We have an algebra isomorphism

$$(C \bowtie_W D)^* \cong C^* \#_{W^*} D^*.$$

Definition 3.4 Let C and D be k -coalgebras and $W : C \otimes D \rightarrow D \otimes C$ a k -linear map. W is called left conormal if

$$(LCN) \quad (I_D \otimes \varepsilon_C)W(c \otimes d) = \varepsilon_C(c)d$$

for all $c \in C, d \in D$. W is called right conormal if

$$(RCN) \quad (\varepsilon_D \otimes I_C)W(c \otimes d) = \varepsilon_D(d)c$$

for all $c \in C$ and $d \in D$. We call W conormal if W is left and right conormal.

Theorem 3.5 Let C and D be k -coalgebras. For a k -linear map $W : C \otimes D \rightarrow D \otimes C$, the following statements are equivalent.

1. $C \bowtie_W D$ is a smash coproduct.
2. The following conditions hold:
 - (CN) W is conormal;
 - (CO) the following octogonal diagram is commutative

$$\begin{array}{ccccc}
 C \otimes D \otimes D & \xrightarrow{W \otimes I_D} & D \otimes C \otimes D & \xrightarrow{I_D \otimes \Delta_C \otimes I_D} & D \otimes C \otimes C \otimes D \\
 \uparrow I_C \otimes \Delta_D & & & & \downarrow I_D \otimes I_C \otimes W \\
 C \otimes D & & & & D \otimes C \otimes D \otimes C \\
 \downarrow \Delta_C \otimes I_D & & & & \uparrow W \otimes I_D \otimes I_C \\
 C \otimes C \otimes D & \xrightarrow{I_C \otimes W} & C \otimes D \otimes C & \xrightarrow{I_C \otimes \Delta_D \otimes I_C} & C \otimes D \otimes D \otimes C
 \end{array}$$

3. The following conditions hold

(CN) W is conormal;

(CP) the following two pentagonal diagrams are commutative:

(CP1)

$$\begin{array}{ccccc}
 C \otimes D & \xrightarrow{W} & D \otimes C & \xrightarrow{\Delta_D \otimes I_C} & D \otimes D \otimes C \\
 \downarrow I_C \otimes \Delta_D & & & & \uparrow I_D \otimes W \\
 C \otimes D \otimes D & \xrightarrow{W \otimes I_D} & D \otimes C \otimes D & &
 \end{array}$$

(CP2)

$$\begin{array}{ccccc}
 C \otimes D & \xrightarrow{W} & D \otimes C & \xrightarrow{I_D \otimes \Delta_C} & D \otimes C \otimes C \\
 \downarrow \Delta_C \otimes I_D & & & & \uparrow W \otimes I_C \\
 C \otimes C \otimes D & \xrightarrow{I_C \otimes W} & C \otimes D \otimes C & &
 \end{array}$$

Definition 3.6 Let C and D be k -coalgebras, and $W : C \otimes D \rightarrow D \otimes C$ a k -linear map. W is called left (resp. right) comultiplicative if (CP1) (resp. (CP2)) is commutative, or, equivalently,

$$\sum ({}^W d)_{(1)} \otimes ({}^W d)_{(2)} \otimes {}^W c = \sum {}^W d_{(1)} \otimes {}^w d_{(2)} \otimes {}^w ({}^W c) \quad (14)$$

respectively

$$\sum {}^W d \otimes ({}^W c)_{(1)} \otimes ({}^W c)_{(2)} = \sum {}^w ({}^W d) \otimes {}^w c_{(1)} \otimes {}^W c_{(2)} \quad (15)$$

for all $c \in C$ and $d \in D$. W is called multiplicative if W is at once left and right multiplicative.

Remark 3.7 Let $W : C \otimes D \rightarrow D \otimes C$ be conormal and comultiplicative. Then the maps

$$\begin{aligned}
 p_C & : C_{W \ltimes D} \rightarrow C, & p_C(c_{W \ltimes D}) &= \varepsilon_D(d)c \\
 p_D & : C_{W \ltimes D} \rightarrow D, & p_D(c_{W \ltimes D}) &= \varepsilon_C(c)d
 \end{aligned}$$

are coalgebra maps. Moreover, the k -linear map

$$\eta = (p_C \otimes p_D) \Delta_{C_{W \ltimes D}} : C_{W \ltimes D} \rightarrow C \otimes D$$

given by

$$\eta(c_{W \ltimes D}) = c \otimes d$$

is bijective, and W can be recovered from η as follows:

$$W = (p_D \otimes p_C) \circ \Delta_{C_{W \ltimes D}} \circ \eta^{-1}$$

Theorem 3.8 Let C , D and Y be k -coalgebras. The following conditions are equivalent.

- 1) There exists a coalgebra isomorphism $Y \cong C_{W \ltimes D}$, for some $W : C \otimes D \rightarrow D \otimes C$;
- 2) Y factorises as $Y = CD$.

Assume that η is bijective. From the proof of Theorem 3.8, it follows that

$$W : C \otimes D \rightarrow D \otimes C, \quad W = (p_D \otimes p_C) \circ \Delta_Y \circ \eta^{-1}$$

is conormal and commultiplicative and

$$\eta : Y \rightarrow C_W \bowtie D$$

is a isomorphism of coalgebras.

The smash coproduct satisfies the following universal property.

Proposition 3.9 *Let C and D be k -coalgebras and $W : C \otimes D \rightarrow D \otimes C$ a conormal and comultiplicative k -linear map.*

Let Y be a coalgebra, and $u : Y \rightarrow C$, $v : Y \rightarrow D$ coalgebra maps such that

$$(v \otimes u) \circ \Delta_Y = W \circ (u \otimes v) \circ \Delta_Y \quad (16)$$

then there exists a unique coalgebra map $w : Y \rightarrow C_W \bowtie D$ such that $p_C w = u$ and $p_D w = v$:

$$\begin{array}{ccccc}
 & & C_W \bowtie D & & \\
 & \swarrow p_C & \uparrow & \searrow p_D & \\
 C & & & & D \\
 & \nwarrow u & \downarrow w & \nearrow v & \\
 & & Y & &
 \end{array}$$

4 Factorisations of algebras and coalgebras

Let us consider now the problem of the factorisation of a bialgebra into algebras and coalgebras. We will call this problem the *bialgebra factorisation problem*.

Definition 4.1 *Let L and H be k -algebras with unit which are also coalgebras with counit and let K be a bialgebra. We say that K factorises as $K = LH$ if we have maps*

$$\begin{array}{ccccc}
 L & \xrightleftharpoons[i_L]{p_L} & K & \xrightleftharpoons[p_H]{i_H} & H
 \end{array}$$

such that

1. i_L and i_H are algebra maps;
2. p_L and p_H are coalgebra maps;
3. the k -linear map

$$\zeta : L \otimes H \rightarrow K, \quad \zeta = m_K \circ (i_L \otimes i_H)$$

is bijective and the inverse is given by

$$\zeta^{-1} : K \rightarrow L \otimes H, \quad \zeta^{-1} = (p_L \otimes p_H) \circ \Delta_K.$$

Let H and L be two algebras that are also coalgebras, and consider two k -linear maps $R : H \otimes L \rightarrow L \otimes H$ and $W : L \otimes H \rightarrow H \otimes L$. $L {}_W \bowtie_R H$ will be equal to $L \#_R H$ as a (not necessarily associative) k -algebra (see Section 2) and equal to $L {}_W \bowtie_R H$ as a (not necessarily coassociative) k -coalgebra (see Section 3).

Definition 4.2 *Let H , L , R and W be as above. We say that $L {}_W \bowtie_R H$ is a smash biproduct of L and H if $L \#_R H$ is a smash product, $L {}_W \bowtie_R H$ is a smash coproduct, and $L {}_W \bowtie_R H$ is a bialgebra.*

Our interest in such structures is motivated by the existing constructions in the Hopf algebra theory: Radford's biproducts ([20]), Takeuchi's bismash product ([27]) and Majid's double crossed-products and bicrossedproducts ([13]). As expected, bialgebra factorisations will be described by the smash biproduct. Therefore, the above mentioned constructions will become examples of biproducts.

Examples 4.3 1) Take a bialgebra H , and let L be at once an algebra in ${}_H \mathcal{M}$ (an H -module algebra), and a coalgebra in ${}^H \mathcal{M}$ (an H -comodule coalgebra). Consider the product $L \times H$ defined by Radford in [20, Theorem 1]. From Example 2.4 3) (with $A = L$ and $D = H$) and Example 3.3 2) (with $D = L$ and $C = H$), it follows that Radford's product is a smash biproduct.

2) In [13], Majid introduced the so-called bicrossproduct of two Hopf algebras, generalizing the bismash product of Takeuchi [27]. We will show that Majid's bicrossproduct is an example of smash biproduct.

Let H and A be two Hopf algebras such that A is a right H -module algebra and H is a left A -comodule coalgebra. The structure maps are denoted as follows

$$\begin{aligned} \alpha : A \otimes H &\rightarrow A \quad ; \quad \alpha(a \otimes h) = a \cdot h \\ \beta : H &\rightarrow A \otimes H \quad ; \quad \beta(h) = \sum h_{<-1>} \otimes h_{<0>} \end{aligned}$$

Now consider the k -linear maps

$$\begin{aligned} R : A \otimes H &\rightarrow H \otimes A \quad ; \quad R(a \otimes h) = \sum h_{(1)} \otimes a \cdot h_{(2)} \\ W : H \otimes A &\rightarrow A \otimes H \quad ; \quad W(h \otimes a) = \sum h_{<-1>} a \otimes h_{<0>} \end{aligned}$$

Then $H {}_W \bowtie_R A$ is nothing else but Majid's bicrossproduct $H {}^\beta \bowtie_\alpha A$.

3) Majid's double crossproduct [13, Section 3.2] is also an smash biproduct. To simplify notation, let us make this clear for the non-twisted version, called bicrossed product in [10].

Let (X, A) be a matched pair of bialgebras. This means that X is a left A -module coalgebra, and A is a right X -module coalgebra such that five additional relations hold (we refer to [10, IX.2.2] for full detail). Write

$$\begin{aligned} \alpha : A \otimes X &\rightarrow X \quad ; \quad \alpha(a \otimes x) = a \cdot x \\ \beta : A \otimes X &\rightarrow A \quad ; \quad \beta(a \otimes x) = a^x \end{aligned}$$

for the structure maps, and take

$$\begin{aligned} R : A \otimes X &\rightarrow X \otimes A \quad ; \quad R(a \otimes x) = \sum a_{(1)} \cdot x_{(1)} \otimes a_{(2)}^{x_{(2)}} \\ W : X \otimes A &\rightarrow A \otimes X \quad ; \quad W(x \otimes a) = a \otimes x \end{aligned}$$

Then $X {}_\tau \bowtie_R A$ is the double crossproduct of the matched pair (X, A) (see also [10, Theorem IX.2.3]).

We will now give a necessary and sufficient condition for $L_W \bowtie_R H$ to be a double product. A necessary requirement will be that $\Delta_{L_W \bowtie H}$ and $\varepsilon_{L_W \bowtie H}$ are algebra maps, and this will be equivalent to some compatibility relations between R and W . We will restrict attention to the case where L and H are bialgebras (see Theorem 4.6). We will first describe smash biproducts in terms of factorisation structures. Using Theorem 2.11 and Theorem 3.8, we obtain the following.

Theorem 4.4 *Let K be a bialgebra and L, H algebras that are also coalgebras. The following statements are equivalent.*

- 1) *There exists a bialgebra isomorphism $K \cong L_W \bowtie_R H$, for some $R : H \otimes L \rightarrow L \otimes H$ and $W : L \otimes H \rightarrow H \otimes L$;*
- 2) *There is a bialgebra factorisation of K as $K = LH$.*

Remark 4.5 Let us now compare the factorisation structures in Majid's constructions with those presented here.

In [13, Prop. 3.12] (see also [15, Thm. 7.2.3]), it is shown that a Hopf algebra K is a double crossproduct of a matched pair of Hopf algebras (X, A) if and only if there exists a sequence

$$X \begin{array}{c} \xrightarrow{i_X} \\ \xleftarrow{p_X} \end{array} K \begin{array}{c} \xleftarrow{i_A} \\ \xrightarrow{p_A} \end{array} A \quad (17)$$

where

- i_X and i_A are injective Hopf algebra maps;
- p_X and p_A are coalgebra maps;
- The k -linear map

$$\zeta : X \otimes A \rightarrow K, \quad \zeta := m_K \circ (i_X \otimes i_A)$$

is bijective, and its inverse is given by the formula

$$\zeta^{-1} : K \rightarrow X \otimes A, \quad \zeta^{-1} := (p_X \otimes p_A) \circ \Delta_K$$

Secondly, a Hopf algebra K is a bicrossed product of two Hopf algebras X and A if and only if there exists a sequence (17) such that

- i_X and p_X are Hopf algebra maps;
- i_A is an algebra map, p_A is a coalgebra map;
- The k -linear map

$$\zeta : X \otimes A \rightarrow K, \quad \zeta := m_K \circ (i_X \otimes i_A)$$

is bijective and its inverse is given by

$$\zeta^{-1} : K \rightarrow X \otimes A, \quad \zeta^{-1} := (p_X \otimes p_A) \circ \Delta_K$$

We refer to [27, Theorem 1.4] for the abelian case and to [13, Theorem 2.3] for the general case.

In a similar way, we can describe the factorisation structures associated to Radford's product (see [20, Theorem 2]). Thus, the double crossed product and bicrossedproduct constructions are completely classifying some particular types of bialgebra factorisation structures.

We will now present necessary and sufficient conditions for $L_W \bowtie_R H$ to be a smash biproduct. For technical reasons, we restricted attention to the situation where H and L are bialgebras.

Theorem 4.6 *Let H and L be bialgebras. For two k -linear maps $R : H \otimes L \rightarrow L \otimes H$ and $W : L \otimes H \rightarrow H \otimes L$, the following statements are equivalent:*

1) $L_W \bowtie_R H$ is a double product;

2) the following conditions hold:

(DP1) R is normal and two-sided multiplicative;

(DP2) W is conormal and two-sided commultiplicative;

(DP3) $(\varepsilon_L \otimes \varepsilon_H)R = \varepsilon_H \otimes \varepsilon_L$;

(DP4) $W(1_L \otimes 1_H) = 1_H \otimes 1_L$;

(DP5) $W(l \otimes h) = W(l \otimes 1_H)W(1_L \otimes h)$;

(DP6) $\sum l_{(1)} l'_{(1)} \otimes^W 1_H \otimes^W [l_{(2)} l'_{(2)}] = \sum l_{(1)} {}^R l'_{(1)} \otimes^R [{}^W 1_H]^U 1_H \otimes^W l_{(2)} {}^U l'_{(2)}$;

(DP7) $\sum^W [h_{(1)} h'_{(1)}] \otimes^W 1_L \otimes h_{(2)} h'_{(2)} = \sum^W h_{(1)} {}^U h'_{(1)} \otimes^W 1_L^R ({}^U 1_L) \otimes^R h_{(2)} h'_{(2)}$;

(DP8) $\sum ({}^R l)_{(1)} \otimes^W [({}^R h)_{(1)}] \otimes^W [({}^R l)_{(2)}] \otimes ({}^R h)_{(2)} = \sum^R l_{(1)} \otimes^R [{}^W h_{(1)}]^U 1_H \otimes^W 1_L {}^r ({}^U l_{(2)}) \otimes^r$

$h_{(2)}$,

for all $l, l' \in L, h, h' \in H$, where $r = R$ and $U = W$.

Proof: From Theorems 2.6 and 3.5, it follows that $L_W \bowtie_R H$ is at once an associative algebra with unit and a coassociative algebra with counit if and only if (DP1) and (DP2) hold.

Furthermore, $\varepsilon_{L_W \bowtie H}$ is an algebra map if and only if (DP3) holds, and (DP4) is equivalent to

$$\Delta_{L_W \bowtie H}(1_H \bowtie 1_L) = (1_L \bowtie 1_H) \otimes (1_L \bowtie 1_H)$$

The proof will be finished if we can show that

$$\Delta_{L_W \bowtie H}(xy) = \Delta_{L_W \bowtie H}(x) \Delta_{L_W \bowtie H}(y) \quad (18)$$

for all $x, y \in L_W \bowtie_R H$ if and only if (DP5-DP8) hold. It is clear that (18) holds for all $x, y \in L_W \bowtie_R H$ if and only if it holds for all

$$x, y \in \{l \bowtie 1_H \mid l \in L\} \cup \{1_L \bowtie h \mid h \in H\}$$

Now (18) holds for $x = l \bowtie 1_H$ and $y = 1_L \bowtie h$ if and only if

$$\sum l_{(1)} \otimes^W h_{(1)} \otimes^W l_{(2)} \otimes h_{(2)} = \sum l_{(1)} \otimes^W 1_H {}^U h_{(1)} \otimes^W l_{(2)} {}^U 1_L \otimes h_{(2)}$$

Applying $\varepsilon_L \otimes I_H \otimes I_L \otimes \varepsilon_H$ to both sides, we see that this condition is equivalent to (DP5).

(18) for all $x = l \bowtie 1_H$ and $y = l' \bowtie 1_H$ is equivalent to (DP6), (18) for all $x = 1_L \bowtie h$ and $y = 1_L \bowtie h'$ is equivalent to (DP7) and (18) for all $x = 1_L \bowtie h$ and $y = l \bowtie 1_H$ is equivalent to (DP8). \square

Let H and L be bialgebras, and consider a k -linear map $R : H \otimes L \rightarrow L \otimes H$. We let $W = \tau_{L,H} : L \otimes H \rightarrow H \otimes L$ be the switch map. We call

$$L_W \bowtie_R H = L \bowtie_R H$$

the R -smash product of H and L . Theorem 4.6 takes the following more elegant form.

Corollary 4.7 *Let H and L be bialgebras. For a k -linear map $R : H \otimes L \rightarrow L \otimes H$, the following statements are equivalent:*

1) $L \bowtie_R H$ is an R -smash product;

2) R is a normal, multiplicative and a coalgebra map.

Futhermore, if H and L have antipodes S_H and S_L and

$$R\tau_{L,H}(S_H \otimes S_H)R\tau_{L,H} = S_L \otimes S_H,$$

then $L \bowtie_R H$ has an antipode given by the formula

$$S_{L \bowtie_R H}(l \bowtie h) = \sum^R S_L(l) \bowtie^R S_H(h).$$

for all $l \in L$, $h \in H$.

Proof: The switch map $W = \tau_{L,H}$ always satisfies equations (DP4-DP7), and (DP3) and (DP8) are equivalent to $R : H \otimes L \rightarrow L \otimes H$ being a coalgebra map. \square

Example 4.8 Let H be a finite dimensional Hopf algebra and $L = H^{*\text{cop}}$, and consider the map

$$R : H \otimes H^{*\text{cop}} \rightarrow H^{*\text{cop}} \otimes H; \quad R(h \otimes h^*) = \sum \langle h^*, S^{-1}(h_{(3)}) \rangle h_{(1)} \otimes h_{(2)}$$

In [15], this map R is called the *Schrödinger operator* associated to H . A routine computation shows that R satisfies the condition of Corollary 4.7, and the R -Smash product $H^{*\text{cop}} \bowtie_R H$ is nothing else then the Drinfel'd Double $D(H)$ in the sense of Radford [21].

The dual situation is also interesting: let H and L be bialgebras, and take the switch map $R = \tau_{H,L} : H \otimes L \rightarrow L \otimes H$. Now we call

$$L_W \bowtie_R H = L_W \bowtie H$$

the W -smash coproduct of L and R , and Theorem 4.6 takes the following form.

Corollary 4.9 *Let H and L be bialgebras. For a k -linear map $W : L \otimes H \rightarrow H \otimes L$, the following statements are equivalent.*

- 1) $L_W \bowtie R$ is a W -smash coproduct;
- 2) W is conormal, comultiplicative and an algebra map.

Proof: For $R = \tau_{H,L}$ the conditions (DP6-DP8) are equivalent to

$$\begin{aligned} W(l'l' \otimes 1_H) &= W(l \otimes 1_H)(W(l' \otimes 1_H)) \\ W(1_L \otimes hh') &= W(1_L \otimes h)W(1_L \otimes h') \\ W(l \otimes h) &= W(1_L \otimes h)W(l \otimes 1_H) \end{aligned}$$

for all $l, l' \in L$, $h, h' \in H$. This three equations together with (DP4) and (DP5) are equivalent to W being an algebra map. \square

Doi [9] and Koppinen [11] introduced the category of *unified Hopf modules* or *Doi-Hopf modules* $\mathcal{M}(H)_A^C$. Some categories that are quite distinct at first sight appear as special cases of this category: the category of classical Hopf modules \mathcal{M}_H^H (see [9]), the category of Yetter-Drinfel'd modules \mathcal{YD}_H^H (see [8]), the category of Long dimodules \mathcal{L}_H^H (see [16]), and many others. We will now present an alternative way to unify these categories.

Definition 4.10 Let H be a Hopf algebra and $R : H \otimes H \rightarrow H \otimes H$ a k -linear map. A twisted R -Hopf module is a k -module M that is at once a right H -module and a right H -comodule such that the following compatibility relation holds:

$$\rho_M(m \cdot h) = \sum m_{<0>} {}^R h_{(1)} \otimes {}^R m_{<1>} h_{(2)} \quad (19)$$

for all $m \in M$, $h \in H$.

The category of twisted R -Hopf module and H -linear H -colinear map will be denoted by $\mathcal{M}(R)_H^H$.

Examples 4.11 1) Let $R = \tau_{H,H}$ be the switch map. Then $\mathcal{M}(\tau_{H,H})_H^H$ is just the category of Hopf modules, as defined in Sweedler's book [24].

2) Let H be a Hopf algebra with bijective antipode and consider the map $R : H \otimes H \rightarrow H \otimes H$ given by the formula

$$R(h \otimes g) = \sum g_{(2)} \otimes S^{-1}(g_{(1)})h$$

for all $h, g \in H$. Then the category $\mathcal{M}(R)_H^H$ is the category \mathcal{YD}_H^H of Yetter-Drinfel'd modules (see [30] and [23]).

3) Let $R : H \otimes H \rightarrow H \otimes H$ be given by the formula

$$R(h \otimes g) = \varepsilon(g)1_H \otimes h$$

for all $h, g \in H$. Then the category $\mathcal{M}(R)_H^H$ is the category \mathcal{L}_H^H of Long H -dimodules defined by Long in [12].

Remark 4.12 Definition 4.10 can be generalized even further. The idea is to replace the map R by a k -linear map $\psi : C \otimes A \rightarrow A \otimes C$, where C is a coalgebra, and A is an algebra. If ψ satisfies certain natural conditions, then the triple (A, C, ψ) is called an *entwining structure* (see [5]), and one can introduce the category $\mathcal{M}(\psi)_A^C$ of *entwining modules*. We refer to [4] for full detail.

Remark 4.13 There is a close relationship between entwining structures and factorisation structures (see [5, Prop. 2.7]): if the coalgebra in an entwining structure is finite dimensional, then there is a one-to-one correspondence between entwining and factorisation structures. Therefore, the Example 2.12 3) gives a complete classification of entwining structures (kC_2, kC_2^*, ψ) .

5 Examples

In [2], a large class of pointed Hopf algebras is constructed, using iterated Ore extensions. In this Section, we will see that an important subclass of this class of Hopf algebras can be viewed as smash biproducts of a group algebra, and a vector space that is an algebra and a coalgebra (but not a bialgebra).

We will use the notation introduced in [7]. Let k be an algebraically closed field of characteristic 0, C a finite abelian group, $C^* = \text{Hom}(C, U(k))$ its character group, and t a positive integer. Assume that the following data are given:

$$\begin{aligned} g &= (g_1, \dots, g_t) \in C^t \\ g^* &= (g_1^*, \dots, g_t^*) \in C^{*t} \\ n &= (n_1, \dots, n_t) \in \mathbf{N}^t \end{aligned}$$

Assume furthermore that $\langle g_l^*, g_l \rangle$ is a primitive n_l -th root of unity, and that $\langle g_r^*, g_l \rangle = \langle g_l^*, g_r \rangle$, for all $r \neq l$. A pointed Hopf algebra

$$K = H(C, n, g^*, g^{-1}, 0)$$

is then given by the following data:

$$x_j c = \langle g_j^*, c \rangle c x_j \quad ; \quad x_j x_k = \langle g_j^*, g_k \rangle x_k x_j \quad ; \quad x_j^{n_j} = 0 \quad (20)$$

$$\varepsilon(c) = 1 \quad ; \quad \Delta(c) = c \otimes c \quad (21)$$

$$\varepsilon(x_i) = 0 \quad ; \quad \Delta(x_i) = x_i \otimes g_i + 1 \otimes x_i \quad (22)$$

$$S(c) = c^{-1} \quad ; \quad S(x_i) = -x_i g_i^{-1} \quad (23)$$

for all $c \in C$ and $i \in \{1, \dots, t\}$. At first glance, (20-23) are not completely the same as (3-8) in [7]. We recover (3-8) in [7] after we replace x_i by $g_i^{-1} x_i = y_i$.

Theorem 5.1 *With notation as above, the Hopf algebra $K = H(C, n, g^*, g^{-1}, 0)$ is a smash biproduct of the group algebra $L = kC$, and an algebra H that is also a coalgebra.*

Proof: We will apply Theorem 4.4. Let H be the subalgebra of K generated by x_1, \dots, x_n :

$$H = k\langle x_1, \dots, x_n \mid x_j^{n_j} = 0 \text{ and } x_j x_k = \langle g_j^*, g_k \rangle x_k x_j \text{ for all } j, k = 1, \dots, t \rangle$$

For $m = (m_1, \dots, m_t) \in \mathbf{N}^t$, we write

$$x^m = x_1^{m_1} \cdots x_t^{m_t}$$

Then

$$\{x^m \mid 0 \leq m_j < n_j, j = 1, \dots, t\} \quad \text{is a basis for } H; \quad (24)$$

$$\{cx^m \mid 0 \leq m_j < n_j, j = 1, \dots, t, c \in C\} \quad \text{is a basis for } H; \quad (25)$$

Now H and L are subalgebras of K , and the inclusions

$$i_H : H \rightarrow K \quad ; \quad i_L : L \rightarrow K$$

are algebra maps. Also the map

$$p_L : K \rightarrow L \quad ; \quad p_L(cx^m) = \varepsilon(x^m)c$$

is a coalgebra map. Consider the maps

$$\begin{aligned} p_H : K &\rightarrow H \quad ; \quad p_H(cx^m) = x^m \\ \Delta_H : H &\rightarrow H \otimes H \quad ; \quad \Delta_H(x^m) = (p_H \otimes p_H)(\Delta_K(x^m)) \\ \varepsilon_H : H &\rightarrow k \quad ; \quad \varepsilon_H(x^m) = \delta_{m0} \end{aligned}$$

We now claim that

$$\Delta_H \circ p_H = (p_H \otimes p_H) \circ \Delta_K \quad (26)$$

For all $c \in C$ and $m \in \mathbf{N}^t$, we have

$$\Delta_H(p_H(cx^m)) = \Delta_H(x^m) = (p_H \otimes p_H)(\Delta_K(x^m))$$

If we write

$$\Delta_K(x^m) = \sum_{d,e \in C} \sum_{r,s \in \mathbf{N}^t} \alpha_{ders} dx^r \otimes ex^s$$

then

$$\Delta_K(cx^m) = \sum_{d,e \in C} \sum_{r,s \in \mathbf{N}^t} \alpha_{ders} cdx^r \otimes cex^s$$

and

$$(p_H \otimes p_H)(\Delta_K(x^m)) = (p_H \otimes p_H)(\Delta_K(cx^m)) = \sum_{d,e \in C} \sum_{r,s \in \mathbf{N}^t} \alpha_{ders} x^r \otimes x^s$$

proving (26). It is clear that p_H is surjective, and it follows from (26) that Δ_H is coassociative, and that p_H is a coalgebra map. Conditions 1) and 2) of Theorem 4.4 are satisfied, and condition 3) remains to be checked. Observe that

$$\zeta(c \otimes x^m) = cx^m$$

and

$$\eta(cx^m) = (p_L \otimes p_H)\Delta_K(cx^m)$$

Now

$$\Delta_K(cx^m) = (c \otimes c) \prod_{i=1}^t (x_i \otimes g_i + 1 \otimes x_i)^{m_i} \quad (27)$$

If we multiply out (27), and apply $p_L \otimes p_H$ to both sides, then all terms are killed in the first factor by p_L , except

$$(c \otimes c) \prod_{i=1}^t (1 \otimes x_i^{m_i}) = c \otimes cx^m$$

and it follows that

$$\eta(cx^m) = (p_L \otimes p_H)(c \otimes cx^m) = c \otimes x^m$$

and η is the inverse of ζ , as needed. \square

Example 5.2 Radford's four parameter Hopf algebras

Let n, N, ν be positive integers such that n divides N and $1 \leq \nu < N$, and let q be a primitive n -th root of unity. In [22], Radford introduced a Hopf algebra $H_{n,q,N,\nu}$, related to invariants for Ribbon Hopf algebras. This Hopf algebra is a special case of the Ore extension construction of [2], in fact

$$K = H_{n,q,N,\nu} = H(C_N, r, g^*, g^{-\nu}, 0)$$

where g is a generator of the cyclic group C_N of order N , $t = 1$, r is the order of q^ν in k and $g^* \in C_N^*$ is defined by $\langle g^*, g \rangle = q$. If we take $\nu = 1$ and $r = n = N$, then we find the Taft Hopf algebra of dimension n^2 (see [25]), and Sweedler's four dimensional Hopf algebra (see [24]) if $n = 2$. Now $H = k[x]/(x^2)$, and the comultiplication Δ_K is given by

$$\Delta_K(g^l x^m) = \sum_{i=0}^m \binom{m}{i}_{q^\nu} g^l x^i \otimes g^{l+\nu i} x^{m-i} \quad (28)$$

where we used the q -binomial coefficients and the q -version of Newton's binomium (see [10, Prop. IV.2.1]). The comultiplication on H takes the form

$$\Delta_H(x^m) = \sum_{i=0}^m \binom{m}{i}_{q^\nu} x^i \otimes x^{m-i} \quad (29)$$

and the counit is given by $\varepsilon_H(x^m) = \delta_{m0}$. The maps R and W can be described explicitly. Using Theorem 2.11 and the fact that $xg = qgx$, we find that $R : H \otimes L \rightarrow L \otimes H$ is given by

$$R(x^m \otimes g^l) = q^{lm} g^l \otimes x^m$$

and, using Theorem 3.8 and (28),

$$W : L \otimes H \rightarrow H \otimes L \ ; \ W(g^l \otimes x^m) = x^m \otimes g^{l+\nu m}$$

Example 5.3 The main result of [6] is that, over an algebraically closed field of characteristic zero, there exists exactly one isomorphism class of pointed Hopf algebras with coradical kC_2 , represented by

$$E(n) = H(C_2, \underline{2}, \underline{g}^*, \underline{g}, 0)$$

where g and g^* are the nonzero elements of C_2 and its dual, and $\underline{2}, \underline{g}^*, \underline{g}$ are the t -tuples with constant entries $2, g^*, g$. In $E(n)$, we have the following (co)multiplication rules:

$$\begin{aligned} x_i g &= -g x_i & x_i x_j &= -x_j x_i \\ \Delta(x_i) &= x_i \otimes g + 1 \otimes x_i & S(x_i) &= -x_i g \end{aligned}$$

Now

$$H = k\langle x_1, \dots, x_t \mid x_i^2 = 0 \text{ and } x_i x_j = -x_j x_i \text{ for } i, j = 1, \dots, t \rangle$$

with comultiplication

$$\begin{aligned} \Delta_H(1) &= 1 \otimes 1 & \Delta_H(x_i) &= 1 \otimes x_i + x_i \otimes 1 \\ \Delta(x_i x_j) &= x_i x_j \otimes 1 + 1 \otimes x_i x_j + x_i \otimes x_j - x_j \otimes x_i \end{aligned}$$

A little computation based on Theorems 2.11 and 3.8 shows that the maps R and W are given by the formulas

$$\begin{aligned} R(h \otimes 1) &= 1 \otimes h & R(1 \otimes g) &= g \otimes 1 \\ R(x_i \otimes g) &= -g \otimes x_i & R(x_i x_j \otimes g) &= g \otimes x_i x_j \\ W(1 \otimes h) &= h \otimes 1 & W(g \otimes 1) &= 1 \otimes g \\ W(g \otimes x_i) &= x_i \otimes 1 & W(g \otimes x_i x_j) &= x_i x_j \otimes g. \end{aligned}$$

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